A possibilistic approach to risk premium

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Abstract

Risk aversion is one of the main themes in risk theory. Risk theory is treated usually by probability theory. The aim of this paper is to propose an approach of the risk aversion by possibility theory initiated by Zadeh in 1978 as an alternative of probability theory in the modeling of uncertain situations. The main notions studied in this paper are the possibilistic risk premium and the possibilistic relative risk premium associated with a fuzzy number A and a utility function u. They reflect the risk aversion of an agent faced with an uncertain situation characterized by a fuzzy number. Under the hypothesis that the utility function u verifies certain hypotheses, one proves a formula which evaluates the possibilistic risk premium and the possibilistic relative risk premium in terms of Arrow-Pratt index and two possibilistic indicators (the expected value and a strong variance of a fuzzy number). Another result is possibilistic Pratt theorem for comparing the risk aversion of two agents (represented by two utility functions u_1 and u_2).

Keywords: risk aversion, risk premium, possibility theory

1 Introduction

Risk theory is developed traditionally in the context of probability theory. It can be successfully applied in case of events which occur with big frequency. The evaluations and forecastings realized with probabilistic risk are efficient when we have a sufficiently large set of data.

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Possibility theory initiated by Zadeh in [24] is an alternative to probability theory in the treatment of uncertainty. It treats those situations of uncertainty in which the events do not occur a large number of times and therefore the information is not extracted from a large volume of data. The development of possibility theory is due to a large number of authors, especially to Dubois and Prade [9]. It has been successfully applied in decision making problems in conditions of uncertainty [3], in fuzzy cooperative games [20], fuzzy neural networks [12], etc.

Possibility theory is based on new concepts such as possibility measure, necessity measure, possibilistic distributions, etc. Traditionally, possibilistic distributions are interpreted as fuzzy sets.

The transition from the probabilistic models to possibilistic models assumes in general two steps:

–instead of random variables there are taken possibilistic distributions;

–the classical concepts related to random variables (mean, variance, covariance, etc.) should be replaced with appropriate possibilistic concepts.

Fuzzy numbers represent an important class of possibilistic distributions. They generalize the real numbers and by Zadeh's extension principle [25] the operations with real numbers are extended to fuzzy numbers [8]. The operations with fuzzy numbers have good arithmetical properties [8].

Among the possibilistic indicators, the mean value and the variance play a central role. The elaboration of the expected utility theory (=EU theory), in particular the probabilistic risk theory [22] is based on them.

The definition and the study of the notions of mean value and variance in a possibilistic context have been tackled by several authors. One of the first contributions in this direction was the introduction by Dubois and Prade in [8] of the interval–valued expectation of a fuzzy number. In [4] Carlsson and Fullér have defined the mean value $E(A)$ of a fuzzy number A and in [11] Fullér and Majlender have defined the weighted possibilistic mean $E_f(A)$ of A. At the same time with the mean value, in these papers there has been introduced the corresponding notion of possibilistic variance: $Var(A)$ in [4] and $Var_f(A)$ in [11]. These indicators of fuzzy numbers have a good mathematical theory, which led to their application in multicriterial decision making problems, finance theory, strategic investment planning etc. (see [3], [19]).

Besides these, there exist other approaches of the mean value and variance in closely related context. Paper [18] proposes a definition of the expected value of a fuzzy variable based on the possibilistic notion of credibility measure. The authors apply this expected value in the study of so–called fuzzy expected value models ([18], p. 447). In [6] there was introduced the variance of a fuzzy random variable as a real interval, while paper [9] deals with the computation of the empirical variance of a set of fuzzy intervals. Paper [16] contains a discussion of faster algorithms for computing mean and variance under Dempster-Shafer uncertainty.

Risk aversion is a concept related to the behaviour of consumers and investors under uncertainty. A person can have different attitudes towards risk: risk averse, risk neutral, risk seeking or risk loving. The option of an agent in a decision making problem is influenced by the attitude towards risk. Risk aversion is studied with probabilistic methods in EU–theory (see [17], [22]). The notion of (probabilistic) risk premium measures the aversion. Fundamental contributions in the study of the risk premium have been realized by Arrow [1] and Pratt [21]. They have been the first ones who evaluated the risk aversion by the coefficients of absolute and relative risk aversion.

This paper intends to study the risk aversion of an agent with respect to the situation of uncertainty described by a fuzzy number. The attitude of the agent is described by an utility function.The framework in which we treat risk aversion has three components: a fuzzy number, a utility function and a weighting function.

The main notions studied in this paper are the possibilistic risk premium and the possibilistic relative risk premium. They measure the risk aversion in situations of possibilistic uncertainty.

The paper is organized as follows:

Section 2 presents some elementary definitions on fuzzy numbers.

In Section 3 we recall some basic concepts in the possibility theory:possibility measure, necessity measure, possibilistic distributions, etc.

Section 4 is dedicated to the expected value $E_f(A)$ associated with a fuzzy number and a weighting function f(see [4], [11]. For a utility function g the possibilistic expected utility $E_f(g(A))$ is introduced (see [4], [11]). This can be a first step in a possibilistic EU–theory. We prove some propositions on the possibilistic expected utility $E_f(g(A))$.

Section 5 deals with two notions of variance associated with a fuzzy number A and a weighting function f : the possibilistic variance $Var_f(A)$ introduced in [11] and the strong possibilistic variance $Var_f^*(A)$. The second indicator is distinct from $Var_f(A)$ and will be used in next section in evaluating the possibilistic risk aversion. Two formulae which express $Var_f^*(A)$ in function of $E_f(A)$ and $Var_f(A)$ are proved.

Section 6 contains the main contribution of this paper. The possibilistic risk premium and the possibilistic relative risk premium are defined as measures of risk aversion of an agent w.r.t. a situation of possibilistic uncertainty described by a fuzzy number. These two possibilistic indicators are introduced by identities similar to the ones corresponding probabilistic indicators. The difference consist in the evaluations present in the terms of identities: in the probabilistic case the usual expected utility ([17], [12]) is used, while in the possibilistic case the possibilistic expected utility introduced in Section 4 is used. The main results of this section

(Propositions 6.3 and 6.5) give us formulae by which the possibilistic risk premium and the possibilistic relative risk premium are expressed which respect to $E_f(A)$, $Var^*_f(A)$ and the Pratt index of the utility function.

In Section 7, $Var^*_f(A)$ and the two possibilistic indicators of risk aversion in the case of trapezoidal fuzzy numbers are calculated.

Section 8 contains a Pratt–type theorem [21]. This result shows that the comparison of the risk aversions of two agents represented by two utility functions u_1 and u_2 is equivalent with the comparison of the Arrow-Pratt indices of u_1, u_2.

The paper conclude with a section of concluding remarks in which there are presented open problems and directions of future researches.

2 Fuzzy numbers

We consider a non–empty subset X as the universe of discourse. The elements of X can be interpreted as individuals, states, attributes, alternatives, etc. A subset of X will be called *cris set*. The membership of elements of X to A can be expressed by the characteristic function χ_A of A: $x \in A$ iff $\chi_A(x) = 1$.

Starting from this fact Zadeh introduced in [25] the notion of fuzzy set. A *fuzzy subset* A of X is a function $A : X \to [0,1]$. For any element $x \in X$, the real number $A(x)$ represents *the degree of membership of x to A*. By *fuzzy set* we understand a fuzzy subset of X.

Usually crisp sets are identified with their characteristic functions; in this way, crisp sets are particular cases of fuzzy sets. We denote by $\mathcal{P}(X)$ the family of crisp subsets of X and by $\mathcal{F}(X)$ the family of fuzzy subsets of X. Then $\mathcal{P}(X) \subseteq \mathcal{F}(X)$.

A fuzzy set is *normal* if $A(x) = 1$ for some $x \in X$. If $A \in \mathcal{F}(X)$ then the crisp set $supp(A) = \{x \in X | A(x) > 0\}$ is called the *support* of A.

We consider now that $X = \mathbf{R}$. For any $\gamma \in [0,1]$, the *γ–level set* $[A]^\gamma$ of A is defined by: $[A]^\gamma = \{x \in X | A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^0 = cl(supp(A))$ where $cl(supp(A))$ is the closure of $supp(A)$.

A fuzzy subset A of $\mathbf{R}$ is *convex* if $[A]^\gamma$ is a convex subset of $\mathbf{R}$ for each $\gamma \in [0,1]$. A *fuzzy number* is a normal, convex and upper semi–continuous fuzzy subset A of $\mathbf{R}$. For a fuzzy number A, $[A]^\gamma$ is a compact and convex subset of $\mathbf{R}$ for all $\gamma \in [0,1]$. A *fuzzy number* is a normal, convex and upper semi–continuous fuzzy subset A of $\mathbf{R}$. For a fuzzy number A, $[A]^\gamma$ is a compact and convex subset of $\mathbf{R}$ for all $\gamma \in [0,1]$.

If we denote $a_1(\gamma) = \min[A]^\gamma$, $a_2(\gamma) = \max[A]^\gamma$, then $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.

Fuzzy numbers are generalizations of real numbers. By using Zadeh's extension principle [25] the operations with real numbers can be extended to fuzzy numbers (see [8], [9]).

A *triangular fuzzy number with center* a is a fuzzy number A of the form

$$A(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & \text{if} \quad a - \alpha \leq x \leq a \\ 1 - \frac{x-a}{\beta} & \text{if} \quad a < x \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

The triangular fuzzy number A of this form will be denoted $A = (a, \alpha, \beta)$. We can prove that $[A]^\gamma = [A - (1-\gamma)\alpha, a + (1-\gamma)\beta]$ for any $\gamma \in [0, 1]$.

A *trapezoidal fuzzy number with tolerance interval* $[a, b]$ is defined by

$$A(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & \text{if} \quad a - \alpha \leq x \leq a \\ 1 & \text{if} \quad a < x \leq b \\ 1 - \frac{x-b}{\beta} & \text{if} \quad b < x \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

In this case we denote $A = (a, b, \alpha, \beta)$. We can prove that $[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$ for any $\gamma \in [0, 1]$.

The triangular number $A = (a, \alpha, \beta)$ is a context–dependent description of the fuzzy property "x is approximately equal to a", where α and β define the context. Similarly, the trapezoidal number $A = (a, b, \alpha, \beta)$ gives a context–dependent description of the fuzzy assertion "x is approximately in the interval $[0, 1]$", where α and β define the context (see [19], p. 26 and 27).

3 Possibilistic distributions

Probability theory is the traditional mathematical instrument with which uncertain phenomena are studied. When we talk about events with little frequency or when we do not dispose of a database sufficiently large the probabilistic modeling of uncertainty does not hold any more. In such situations we use the possibility theory of Zadeh [25].

The development of possibility theory was achieved in a big measure by an analogy with probability theory:

• probabilistic concepts have been replaced with possibilistic concepts;

• probabilistic results have been converted into possibilistic results (for this the translation in possibilistic terms of some proofs from probability theory has been tried).

Possibility measure and necessity measure are fundamental concepts in possibility theory. They represent for possibility theory what the notion of probabilistic measure means for probability theory.

Let X be a non–empty set. A *possibility measure* on X is a function $\Pi : \mathcal{P}(X) \to [0, 1]$ such that the following conditions are verified:

$(P1)$ $\Pi(\emptyset) = 0$; $\Pi(X) = 1$;

$(P2)$ $\Pi(\bigcup_{i\in I} D_i) = \sup_{i\in I} \Pi(D_i)$ for any family $\{D_i\}_{i\in I}$ of subsets of X.

A *necessity measure* on X is a function $N : \mathcal{P}(X) \to [0,1]$ such that the following conditions are verified:

$(N1)$ $N(\emptyset) = 0$; $N(X) = 1$;

$(N2)$ $N(\bigcap_{i\in I} D_i) = \inf_{i\in I} N(D_i)$ for any family $\{D_i\}_{i\in I}$ of subsets of X.

Let Π (resp. N) be a possibility measure (resp. a necessity measure) on X. Let us consider the functions:

$\Pi^{nec} : \mathcal{P}(X) \to [0,1]$; $N^{pos} : \mathcal{P}(X) \to [0,1]$

defined by

$\Pi^{nec}(D) = 1 - \Pi(X - D)$ for any $D \subseteq X$;

$N^{pos}(D) = 1 - N(X - D)$ for any $D \subseteq X$.

Proposition 3.1 *[18]*

(i) Π^{nec} *is a necessity measure on* X*;*

(ii) N^{pos} *is a possibility measure on* X*;*

(iii) $(\Pi^{nec})^{pos} = \Pi$ *and* $(N^{pos})^{nec} = N$.

Remark 3.2 *Proposition 3.1 shows that each necessity measure* N *is uniquely determined by the possibility measure* N^{pos} *and viceversa, each possibility measure* Π *is uniquely determined by the necessity measure* Π^{pos}*. Accordingly, it suffices to use only the notion of possibility measure.*

Probability theory studies random variables which together with their indicators (mean value, variance, covariance etc.) describe uncertain situations. In possibility theory, the place of random variables is taken by possibilistic distributions.

A *possibility distribution* on the set is a function $\mu : X \to [0,1]$ such that $\sup_{x\in X} \mu(x) = 1$; μ is said to be *normalized* if $\mu(x) = 1$ for some $x \in X$.

The notion of possibility distribution is tightly connected with that of possibility measure. Now we specify the way the two entities (possibility measure and possibility distribution) determine one another.

Let Π be a possibility measure on X and μ a possibility distribution on X. Let us consider the functions

$\mu_\Pi : X \to [0,1]$, $Pos_\mu : \mathcal{P}(X) \to [0,1]$

defined by

$\mu_\Pi(x) = \Pi(\{x\})$ for any $x \in X$;

$Pos_\mu(D) = \sup_{x\in D} \mu(x)$ for any $D \in \mathcal{P}(X)$.

Proposition 3.3 *(i) μ_Π is a possibility distribution on X;*

(ii) Pos_μ is a possibility measure on X;

(iii) $\mu_{Pos_\mu} = \mu$ and $Pos_{\mu_\Pi} = \Pi$.

Let μ be a possibility distribution on X and μ_Π its possibility measure. Then one can consider the following necessity measure $Nec_\mu : \mathcal{P}(X) \to [0,1]$:

$Nec_\mu(D) = 1 - Pos_\mu(X - D) = 1 - \sup_{x \notin D} \mu(D)$

In this paper we will work only with possibilistic distributions defined on $\mathbf{R}$. Fuzzy numbers represent the most important class of possibilistic distributions on $\mathbf{R}$.

4 Possibilistic expected value

We already mentioned in the previous section that in possibility theory, the place of random variables is taken by possibilistic distributions. A crucial problem is to define some possibilistic notions corresponding to the known possibilistic indicators (mean value, variance, covariance, etc.)

In a certain measure for fuzzy numbers this desideratum has been achieved in a series of papers of Carlsson et al. ([3], [4], [5], [11], [12], [19]). The approaches of these authors is based on the observation that the γ–level sets $[A]^\gamma$ of a fuzzy number A can be written as intervals $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.

The notion of possibilistic mean value of a fuzzy number A was introduced by Carlsson et al. in [3] and has been generalized in [11] by the notion of expected value of A w.r.t a weighting function f.

A function $f : [0,1] \to \mathbf{R}$ is a *weighting function* if it is non–negative, monotone increasing and verifies the normality condition $\int_0^1 f(x)dx = 1$.

We fix a weighting function f and a fuzzy number A such that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.

The *expected value* of A w.r.t. f is defined by

(1) $E_f(A) = \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma$

If $f(\gamma) = 2\gamma$ for all $\gamma \in [0,1]$ then $E_f(A)$ is exactly the possibilistic mean value introduced in [3].

As seen, the definition of $E_f(A)$ is based on the representation of $[A]^\gamma$ as an interval $[a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.

For any $\gamma \in [0,1]$ we denote by X_γ the uniform distribution corresponding to the interval $[a_1(\gamma), a_2(\gamma)]$. X_γ is a random variable whose density $\rho(x)$ has the following expression

$\rho(x) = \begin{cases} \frac{1}{a_2(\gamma)-a_1(\gamma)} & \text{if} \quad x \in [a_1(\gamma), a_2(\gamma)] \\ 0 & \text{if} \quad x \notin [a_1(\gamma), a_2(\gamma)] \end{cases}$

A simple calculation shows that

$E_f(A) = \int_0^1 E(X_\gamma) f(\gamma) d\gamma$

where $E(X_\gamma)$ is the mean value of X_γ (see [15]).

If X is a random variable and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function then $g(X)$ is a random variable, hence we can consider the expected value $E(g(X))$ of $g(X)$. Function g can be considered a utility function and then $E(g(X))$ means the *expected utility* of X w.r.t. g [15].

We pose the question which is the possibilistic correspondent of the expected utility $E(g(X))$: for a fuzzy number A, who is $E_f(g(A))$?

If $g : \mathbf{R} \to \mathbf{R}$ is a continuous function and A a fuzzy number, then $g(A)$ is a fuzzy subset of $\mathbf{R}$ defined by Zadeh's extension principle [25]: for any $y \in \mathbf{R}$, $g(A)(y)$ will be given by

$g(A)(y) = \begin{cases} \sup\limits_{g(x)=y} A(x) & \text{if} \quad x \in [a_1(\gamma), a_2(\gamma)] \\ 0 & \text{if} \quad x \notin [a_1(\gamma), a_2(\gamma)] \end{cases}$

In general, $g(A)$ is not a fuzzy number, therefore $E_f(g(A))$ cannot be defined by formula (1). In [13] the following definition of $E_f(g(A))$ has been proposed:

(2) $E_f(g(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x) dx] f(\gamma) d\gamma$

If $g(x) = x$ for any $x \in \mathbf{R}$ then we find formula (1). In interpretation, $E_f(g(A))$ means *the possibilistic expected utility* of the fuzzy number A w.r.t. the utility function g.

In the following we will establish some properties of $E_f(g(A))$.

Proposition 4.1 *Let A be a fuzzy number, $g : \mathbf{R} \to \mathbf{R}$, $h : \mathbf{R} \to \mathbf{R}$ two continuous functions and $a, b \in \mathbf{R}$. We consider the continuous function $u : \mathbf{R} \to \mathbf{R}$ defined by $u(x) = ag(x) + bh(x)$ for any $x \in \mathbf{R}$. Then*

$E_f(u(A)) = aE_f(g(A)) + bE_f(h(A))$.

Proof. According to (2) one gets

$E_f(u(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(x) dx] f(\gamma) d\gamma =$

$= a \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x) dx] f(\gamma) d\gamma +$

$+ b \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} h(x)] f(\gamma) d\gamma$

$= aE_f(g(A)) + bE_f(h(A))$ ∎

Corollary 4.2 *Let A be a fuzzy number, $g : \mathbf{R} \to \mathbf{R}$ a continuous function and $a, b \in \mathbf{R}$. If $u = ag + b$ then $E_f(u(A)) = aE_f(g(A)) + b$*

Lemma 4.3 *Let $g : \mathbf{R} \to \mathbf{R}$, $h : \mathbf{R} \to \mathbf{R}$ be two continuous functions. If $g \leq h$ then $E_f(g(A)) \leq E_f(h(A))$.*

Proof. Assume $g \leq h$. Then, for any $\gamma \in [0,1]$, we have
$\int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx \leq \int_{a_1(\gamma)}^{a_2(\gamma)} h(x)dx$.
Since $a_1(\gamma) \leq a_2(\gamma)$ and $f \geq 0$, the following inequality holds:
$[\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx]f(\gamma) \leq [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} h(x)dx]f(\gamma)$
for each $x \in [0,1]$. By taking into account (2), one obtains $E_f(g(A)) \leq E_f(h(A))$. ∎

The following result is a possibilistic version of the probabilistic Jensen inequality. Its proof will be an adaptation of the proof in [15], p. 201, but it will use the properties of the possibilistic expected value.

Proposition 4.4 *If the function $g : \mathbf{R} \to \mathbf{R}$ is convex and continuous, then $g(E_f(A)) \leq E_f(g(A))$.*

Proof. From the real analysis we know that if g is convex then there exist two sequences of real numbers (a_n) and (b_n) such that
(a) $g(x) = \sup\limits_n (a_n x + b_n)$, for any $x \in \mathbf{R}$.
Let $n \in \mathbf{N}$. Then $a_n x + b_n \leq g(x)$ for any $x \in \mathbf{R}$. By Lemma 4.3 we get
$E_f(a_n A + b_n) \leq E_f(g(A))$.
Applying Proposition 4.1, $E_f(a_n A + b_n) = a_n E_f(A) + b_n$, hence
(b) $a_n E_f(A) + b_n \leq E_f(g(A))$ for all $n \in \mathbf{N}$.
From (a) and (b) it follows
$g(E_f(A)) = \sup_n (a_n E_f(A) + b_n) \leq E_f(g(A))$. ∎

Corollary 4.5 *If the function $g : \mathbf{R} \to \mathbf{R}$ is concave and continuous then $g(E_f(A)) \geq E_f(g(A))$.*

Proof. It follows from Proposition 4.4 and from the fact that g is concave iff $-g$ is convexe. ∎

5 Two possibilistic notions of variance

If X is a random variable and $E(X)$ is its expected value then the variance $Var(X)$ of X has the following form:

(1) $Var(X) = E[(X - E(X))^2]$

We consider the problem of defining a corresponding notion of variance for possibilistic distributions. In this section we shall discuss two modes of introducing the variance for fuzzy numbers.

We fix a weighting function f and a fuzzy number A. We assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0, 1]$.

The first way of defining the possibilistic variance of A uses the form of (probabilistic) variance corresponding to the uniform distribution of the intervals $[a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. We denote by X_γ the uniform distribution of the interval $[a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0, 1]$.

According to [11], [19] the (possibilistic) variance $Var_f(A)$ of A with respect to f has the following form

(2) $Var_f(A) = \int_0^1 Var(X_\gamma) f(\gamma) d\gamma$

$= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma$

If $f(\gamma) = 2\gamma$ then one obtains the notion of crisp possibilistic variance introduced by Carlsson and Fullér in [3].

The second way of defining the variance of a fuzzy number starts from the form (1) of the probabilistic variance. We consider the continuous function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = (x - E_f(A))^2$ for any $x \in \mathbf{R}$. The *strong* (possibilistic) variance $Var^*_f(A)$ of A w.r.t. f is defined by

(3) $Var^*_f(A) = E_f(g(A))$

According to formula (2) from Section 4:

(4) $Var^*_f(A) = \int_0^1 \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))^2) dx] f(\gamma) d\gamma$.

In the following we investigate the relationship between the indicators $Var_f(A)$ and $Var^*_f(A)$.

Proposition 5.1 $Var^*_f(A) = \frac{1}{3} \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma) a_2(\gamma)] f(\gamma) d\gamma + E_f^2(A)$

Proof. We start from formula (4). One notices that

$\int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))^2 dx = \frac{(a_2(\gamma) - E_f(A))^3 - (a_1(\gamma) - E_f(A))^3}{3}$.

Then (4) becomes

(5) $Var^*_f(A) = \frac{1}{3} \int_0^1 w(\gamma) f(\gamma) d\gamma$

where

$w(\gamma) = (a_1(\gamma) - E_f(A))^2 + (a_2(\gamma) - E_f(A))^2 + (a_1(\gamma) - E_f(A))(a_2(\gamma) - E_f(A))$.

A simple calculation shows that

(6)$w(\gamma) = [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma) a_2(\gamma)] - 3E_f(A)[a_1(\gamma) + a_2(\gamma)] + 3E_f^2(A)$

From formula (1) Section 4 we have

$\int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma = 2E_f(A)$.

At the same time, from the definition of the weighting function, $\int_0^1 f(\gamma)d\gamma = 1$. Then, from relations (5) and (6) we deduce that

$Var_f^*(A) = \frac{1}{3}\int_0^1 [a_1^2(\gamma)+a_2^2(\gamma)+a_1(\gamma)a_2(\gamma)]f(\gamma)d\gamma - \frac{1}{3}E_f(A)\int_0^1 [a_1(\gamma)+a_2(\gamma)]f(\gamma)d\gamma + \frac{1}{3}3E_f^2(A) =$

$= \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma)]f(\gamma)d\gamma - E_f^2(A)$. ∎

Proposition 5.2 $Var_f^*(A) = 4Var_f(A) - E_f^2(A) + \int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma$

Proof. $a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma) = [a_1(\gamma) - a_2(\gamma)]^2 + 3a_1(\gamma)a_2(\gamma)$.
Then, by applying Proposition 5.1 one obtains
$Var_f^*(A) = \frac{1}{3}\int_0^1 [a_1(\gamma) - a_2(\gamma)]^2 f(\gamma)d\gamma + \frac{1}{3}\int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma - E_f^2(A)$.
But, according to (2) $\int_0^1 [a_1(\gamma) - a_2(\gamma)]^2 f(\gamma)d\gamma = 12Var_f(A)$, therefore
$Var_f^*(A) = \frac{1}{3}12Var_f(A) + \frac{1}{3}\int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma - E_f^2(A)$
$= 4Var_f(A) - E_f^2(A) + \int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d(\gamma)$. ∎

Remark 5.3 *According to (2) and (4) we have* $Var_f(A) \geq 0$ *and* $Var_f^*(A) \geq 0$.

Corollary 5.4 $E_f^*(A) \leq 4Var_f(A) + \int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma$

Proof. By Proposition 5.1 and Remark 5.3. ∎

6 Risk aversion through fuzzy numbers

In this section we propose a possibilistic approach to some fundamental concepts of risk aversion. Some notions and results of probability theory of risk aversion are translated into a possibilistic framework.

By using the possibilistic expected value $E_f(A)$ introduced in the previous section, we shall define the notion of *possibilistic risk premium* associated with a fuzzy number, a utility function and a weighting function. This concept is a measure of the risk aversion in situations when the measures subject to risk are modelled by fuzzy numbers.

Arrow [1] and Pratt [21] have established a formula which expresses the probabilistic risk premium depending on the probabilistic version and of a coefficient of absolute risk aversion. We prove a formula which evaluates the possibilistic risk premium in terms of the possibilistic variance $Var_f^*(A)$ and of a coefficient (=the possibilistic coefficient of absolute risk aversion). This formula certifies that in the study of the possibilistic risk aversion, it is indicated to use the possibilistic variance $Var_f^*(A)$ and not the indicator $Var_f(A)$.

In this section there is also defined a second risk indicator called the *relative risk premium* . For this, there is found a calculation formula depending on $Var_f^*(A)$, $E_f(A)$ and on a coefficient of risk aversion.

The proofs of the results of this section are based especially on Proposition 4.1.

We expose first some ideas from the probabilistic risk theory, by following the presentation of [17], Chapter 2 (Measuring Risk Aversion and Risk), but in a slightly modified form.

The framework of the probabilistic risk aversion is assured by a utility function and a random variable. The utility function expresses the attitude of an agent. In this probabilistic setting the risk premium is defined as a measure of the risk aversion of the agent w.r.t. the uncertain situation described by the random variable. [1] Now we shall formalize these ideas.

Let Ω be a set of states endowed with a probability space (Ω, K, P) where K is the σ–algebra on Ω and P is a probability defined on K. The sets of K represent the events, and the function $P : K \to [0, 1]$ evaluates the probability of these events. Assume $X : \Omega \to \mathbf{R}$ is a random variable and $u : \mathbf{R} \to \mathbf{R}$ a continuous function (=utility function). Then $u(X) = u \circ X$ is a random variable and $E(u(X))$ will denote the expected value of $u(X)$. If X has a density function $f : \mathbf{R} \to \mathbf{R}$ then

(1) $E(u(X)) = \int_{-\infty}^{+\infty} u(x)f(x)dx$

Following [17] , p. 15, the *risk premium* ρ_X (associated with the random variable X and the utility function u) is defined by the identity

(2) $E(u(X)) = u(E(X) - \rho_X)$.

In interpretation, the risk premium is "the maximum amount by which the agent is willing to decrease the expected return from the lottery ticket to have a sure return" ([17], p. 19).

Proposition 6.1 *([17], p. 21) Assume that u is twice differentiable, strictly concave and increasing. Then*

(3) $\rho_X = -\frac{1}{2}\sigma_X^2 \frac{u''(E(X))}{u'(E(X))}$

where σ_X^2 is the variance of X.

The Arrow–Pratt index (=the coefficient of absolute risk aversion) associated with a utility function u is introduced by the equality

$r_a(x) = -\frac{u''(x)}{u'(x)}$ for all $x \in \mathbf{R}$.

[1]Usually, we can consider that the random variable represents a lottery.

Proposition 6.1 shows that the risk premium ρ_X can be expressed in terms of the Arrow–Pratt index and of two probabilistic indicators (expected value and variance). Thus the Arrow–Pratt index can be viewed as a measure of the risk aversion of the agent represented by the utility function u.

Starting from these ideas we shall define a notion of possibilistic risk premium.

The context in which we shall define and study this concept will have three components: a weighting function, a fuzzy number (representing a possibilistic distribution) and a utility function (representing an agent).

We fix a weighting function f.

Let A be a fuzzy number and $u : \mathbf{R} \to \mathbf{R}$ be a continuous function. Recall from the previous section the formula

(4) $E_f(u(A)) = \int_0^1 (\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(x)dx) f(\gamma)d\gamma.$

Definition 6.2 *The possibilistic risk premium $\rho_A = \rho_{A,f,u}$ (associated with the fuzzy number A, the weighting function f and the utility function u is defined by*

(5) $u(E_f(A) - \rho_A) = E_f(u(A))$.

The above definition has been obtained starting from equality (2): the random variable X is replaced with the fuzzy number A and the. probabilistic mean value $E(X)$ will be replaced with the possibilistic mean value $E_f(A)$. Then the interpretation of equality (5) is analogous to (2); the difference between them consists in the way the terms of the two equalities are evaluated: in (2) by the probabilistic mean value $E(.)$, while in (5) by the possibilistic mean value $E_f(.)$.

Then the possibilistic risk premium is defined so that the possibilistic expected utility of a gamble equals the utility of the gamble's possibilistic expected value minus its possibilistic risk premium.

The use of the probabilistic risk premium or of the possibilistic risk premium will depend on the classification of the risk situation in a probabilistic model or in a possibilistic model.

Usually we can impose some conditions on the utility function u, from which its injectivity follows (e. g. conditions from Propositions 6.3). In the presence of such conditions, from (5) it follows that the possibilistic risk premium ρ is unique with respect to A and u.

The following result is a possibilistic version of Proposition 6.1.

Proposition 6.3 *Assume that u is twice differentiable, strictly concave and increasing. Then the possibilistic risk premium ρ_A has the form:*

(6) $\rho_A = -\frac{1}{2} Var_f^*(A) \frac{u"(E_f(A))}{u"(E_f(A))}$

Proof. The condition that u is twice differentiable allows us to write, according to the Taylor formula of the second degree:

$u(x) = u(E_f(A)) + u'(E_f(A))(x - E_f(A)) + \frac{u"(E_f(A))}{2}(x - E_f(A))^2 + R_2(x)$.

Ignoring the error term $R_2(x)$ we have:

$u(x) \approx u(E_f(A)) + u'(E_f(A))(x - E_f(A)) + \frac{u"(E_f(A))}{2}(x - E_f(A))^2$.

Let us consider the continuous functions $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ defined by

$g(x) = x - E_f(A)$, $h(x) = (x - E_f(A))^2$, for any $x \in \mathbf{R}$.

If we denote $a = u(E_f(A))$, $b = u'(E_f(A))$ and $c = \frac{1}{2}u"(E_f(A))$, then $u = a + bg + ch$. By Proposition 4.1 we get

$E_f(u(A)) = a + bE_f(g(A)) + cE_f(h(A))$.

We remark that $E_f(h(A)) = Var_f^*(A)$. According to formula (2) of Section 4 we obtain

$E_f(g(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))dx] f(\gamma)d\gamma =$

$= \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} xdx] f(\gamma)d\gamma -$

$-E_f(A) \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} dx] f(\gamma)d\gamma =$

$= \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma - E_f(A) = E_f(A) - E_f(A) = 0$.

Therefore we get the following equality:

(7) $E_f(u(A)) = u(E_f(A)) + \frac{u"(E_f(A))}{2} Var_f^*(A)$.

According to condition (5) and retaining from the Taylor formula for $u(E_f(A) - \rho_A))$ only the first two terms, we have:

(8) $E_f(u(A)) = u(E_f(A) - \rho_A) = u(E_f(A)) - u'(E_f(A))\rho_A$.

From (7) and (8) it follows:

(9) $\rho_A = -\frac{1}{2} Var_f^*(A) \frac{u"(E_f(A))}{u'(E_f(A))}$. ∎

We notice the similitude between the expression (3) of ρ_X and the form (9) of ρ_A: if in (3) we replace σ_X^2 with $Var_f^*(A)$ and $E(X)$ with $E_f(A)$ then we obtain (9).

By analogy with the probabilistic case ([17], p. 21), we shall define the *possibilistic coefficient of absolute risk assessment* at the level of wealth $E_f(A)$ by:

(10) $r_a(E_f(A)) = -\frac{u"(E_f(A))}{u'(E_f(A))}$.

$r_a(E_f(A))$ is called *the possibilistic Arrow–Pratt coefficient of risk aversion* .

It is an indicator which measures the risk aversion of an agent (represented by its utility function u) with respect to an uncertain situation described by a fuzzy number A.

Besides the risk premium ρ_X associated with a random variable X in [17] p. 22 there is introduced another probabilistic risk indicator: the *relative risk premium* $\hat{\rho}_X$ defined by the equality:

(11) $E(u(X)) = u(E(X)(1 - \hat{\rho}_X))$.

This leads to the following possibilistic concept:

Definition 6.4 *The possibilistic relative risk premium* $\hat{\rho}_A = \hat{\rho}_{A,f,u}$ *(associated with the fuzzy number A, the weighting function f and the utility function u) is defined by:*

(12) $E_f(u(A)) = u(E_f(A)(1 - \hat{\rho}_A))$.

Proposition 6.5 *Assume that u is twice differentiable, strictly concave and increasing. Then the possibilistic relative risk premium* $\hat{\rho}_A$ *has the form:*

(13) $\hat{\rho}_A = -\frac{1}{2} Var_f^*(A) E_f(A) \frac{u"(E_f(A))}{u'(E_f(A))}$.

Proof. According to relation (7) from the proof of Proposition 6.3 we have

$E_f(u(A)) = u(E_f(A)) + \frac{u'(E_f(A))}{2} Var_f^*(A)$.

According to (12) and retaining from the Taylor formula for $u(E_f(A) - E_f(A)\hat{\rho})$ only the first two terms, one obtains:

$E_f(u(A)) = u(E_f(A) - E_f(A)\hat{\rho})$

$= u(E_f(A)) - u'(E_f(A))E_f(A)\hat{\rho}_A$.

By equalling the two expressions of $E_f(u(A))$ it follows (13). ∎

We notice that Proposition 4.1 is the main instrument in the proofs of Propositions 6.3 and 6.5.

The *possibilistic coefficient of relative risk aversion* at the level of wealth $E_f(A)$ is defined by

$r_r(E_f(A)) = -E_f(A) \frac{u"(E_f(A))}{u'(E_f(A))}$.

According to Propositions 6.3 and 6.5 from above, in the expression of ρ_A and $\hat{\rho}_A$ appears the possibilistic variance $Var_f^*(A)$. Due to Proposition 5.2, ρ_A and $\hat{\rho}_A$ can be expressed by means of $Var_f(A)$ too.

7 An example

In this section we will compute the possibilistic risk premium ρ_A and the possibilistic relative risk premium $\hat{\rho}_A$ for the case of trapezoidal fuzzy numbers.

Let us consider a triangular fuzzy number $A = (a, b, \alpha, \beta)$.

(1) $A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if} \quad a - \alpha \leq t \leq a \\ 1 & \text{if} \quad a < t \leq b \\ 1 - \frac{t-b}{\beta} & \text{if} \quad b < t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$

Recall that $[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$ for all $\gamma \in [0, 1]$, hence

(2) $a_1(\gamma) = a - (1-\gamma)\alpha$; $a_2(\gamma) = b + (1-\gamma)\beta$.

We assume that $f(\gamma) = 2\gamma$ for any $\gamma \in [0, 1]$. Thus by formula (1) of Section 4 and formula (3) of Section 5:

(3) $E_f(A) = \frac{a+b}{2} + \frac{\beta-\alpha}{6}$

(4) $Var_f(A) = \frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha+\beta)^2}{72}$

A simple calculation shows that

(5) $\int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma = ab + \frac{a\beta-b\alpha}{3} - \frac{\alpha\beta}{6}$.

By taking into account (3), (4) and (5), Proposition 5.2 leads us to the following formula for the calculation of the possibilistic variance $Var_f^*(A)$:

$Var_f^*(A) = 4Var_f(A) - E_f^2(A) + \int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma$

$= \frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha^2+\beta^2)}{36}$

If u is a utility function which verifies the conditions of Proposition 6.3, then the possibilistic risk premium ρ_A will have the following form:

(6) $\rho_A = -\frac{1}{2}[\frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha^2+\beta^2)}{36}]\frac{u"(E_f(A))}{u'(E_f(A))}$,

where $E_f(A) = \frac{a+b}{2} + \frac{\beta-\alpha}{6}$.

According to Proposition 6.5, the possibilistic relative risk premium $\hat{\rho}_A$ will be given by

(7) $\hat{\rho}_A = -\frac{1}{2}[\frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha^2+\beta^2)}{36}][\frac{a+b}{2} + \frac{\beta-\alpha}{6}]\frac{u"(E_f(A))}{u'(E_f(A))}$.

Depending on the form of the utility function we have various expressions of ρ_A and $\hat{\rho}_A$. If we consider $u(x) = -e^{-2x}$ then u verifies the conditions required in Proposition 6.3. One notices that

$\frac{u"(x)}{u'(x)} = -2$ for each $x \in \mathbf{R}$,

therefore in this case ρ_A and $\hat{\rho}_a$ will have the form:

(8) $\rho_A = \frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha^2+\beta^2)}{36}$

(9) $\hat{\rho}_A = [\frac{a+b}{2} + \frac{\beta-\alpha}{6}][\frac{(b-a)^2}{12} + \frac{(b-a)(\alpha+\beta)}{18} + \frac{(\alpha^2+\beta^2)}{36}]$.

Now we shall consider the particular case of a triangular fuzzy number $A = (a, \alpha, \beta)$. Then by (3) and (4) we have

$E_f(A) = a + \frac{\beta-\alpha}{6}$

$Var_f(A) = \frac{(\alpha+\beta)^2}{72}$

and the strong possibilistic variance $Var_f^*(A)$ becomes:

$Var_f^*(A) = \frac{\alpha^2+\beta^2}{36}$.

Then, according to (6) and (7), ρ_A and $\hat{\rho}_A$ gets the form:

$\rho_A = -\frac{1}{2}[\frac{\alpha^2+\beta^2}{36}]\frac{u"(E_f(A))}{u'(E_f(A))}$

$\hat{\rho}_A = -\frac{1}{2}[\frac{\alpha^2+\beta^2}{36}][a + \frac{\beta-\alpha}{6}]\frac{u"(E_f(A))}{u'(E_f(A))}$.

In case when $u(x) = -e^{2x}$ we obtain:

$\rho_A = \frac{\alpha^2+\beta^2}{36}$

$\hat{\rho}_A = [a + \frac{\beta-\alpha}{6}][\frac{\alpha^2+\beta^2}{36}]$.

8 Possibilistic Pratt theorem

We have seen in the previous section that the notion of possibilistic risk premium is a measure of risk aversion of an agent (represented by the utility function u) faced with an uncertain situation (described by a fuzzy number).

If there exist two agents (represented by the utility functions u_1, u_2) we pose the question of comparing their risk aversions faced with the same uncertain situation.

In case of the probabilistic risk, the problem was answered by a known theorem of Pratt ([21]).

In this section we shall prove a Pratt–type theorem [21] for the possibilistic risk aversion associated with a fuzzy number, a utility function and a weighting function (see [14]). By combining this result with the Pratt theorem for the possibilistic risk one reaches a surprising result: the aversion to the probabilistic risk is equivalent with the aversion to the possibilistic risk.

In the following we shall assume that the utility function verifies the properties of Proposition 6.3. We recall that the Arrow–Pratt index associated with the utility function u is defined by

$r_u(x) = -\frac{u''(x)}{u'(x)}$ for each $x \in \mathbf{R}$.

Let u_1, u_2 be two utility functions. Let us denote by $r_1(x) = r_{u_1}(x)$ and $r_2(x) = r_{u_2}(x)$ the Arrow–Pratt indices of u_1 and u_2.

Let $\mathbf{B}$ be the Σ–algebra of the Borelian subsets of $\mathbf{R}$.

Theorem 8.1 *[21] The following assertions are equivalent:*

(1) $r_1(x) \geq r_2(x)$ *for each* $x \in \mathbf{R}$*;*

(2) $u_1 \circ u_2^{-1}$ *is concave;*

(3) $\rho_{X,u_1} \geq \rho_{X,u_2}$ *for any random variable* X *with respect to* $(\mathbf{R}, \mathbf{B})$.

Remark 8.2 *Suppose that agents* 1 *and* 2 *are represented by the utility functions* u_1 *and* u_2*. By taking into account the interpretation of the probabilistic risk premium, condition (3) of Theorem 8.1 means that the agent* 1 *is more risk–prone than the agent* 2*, with respect to any random variable* X*. In this case we shall denote* $u_1 \succeq_{probab} u_2$ *iff the Arrow–Pratt index of* u_1 *is bigger than the Arrow–Pratt index of* u_2*. It follows that the Arrow–Pratt index is a measure of the probabilistic risk aversion.*

The following result is the possibilistic aversion of the Pratt theorem:

Theorem 8.3 *Let* u_1*,* u_2 *be two utility functions and* $r_1 = r_{u_1}$*,* $r_2 = r_{u_2}$ *be the Arrow—Pratt indices of* u_1 *and* u_2*. The following assertions are equivalent:*

(1) $r_1(x) \leq r_2(x)$*, for any* $x \in \mathbf{R}$*;*

(2) $u \circ u_2^{-1}$ *is concave;*

(3) For all fuzzy numbers A and weighting functions f, we have
$\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$.

Proof. (1) $\Leftrightarrow$ (2) By Pratt's theorem:
(2) $\Rightarrow$ (3) Let $\rho_i = \rho_{A,f,u_i}$, $i = 1, 2$. According to Definition 6.2
$u_1(E_f(A) - \rho_1) = E_f(u_1(A))$
$u_2(E_f(A) - \rho_2) = E_f(u_2(A))$.
By applying to these equalities the inverses u_1^{-1}, u_2^{-1} of u_1 and u_2 one deduces:
$\rho_1 = E_f(A) - u_1^{-1}(E_f(u_1(A))$
$\rho_2 = E_f(A) - u_2^{-1}(E_f(u_2(A))$
By subtracting these two inequalities one obtains:
(a) $\rho_1 - \rho_2 = u_2^{-1}(E_f(u_2(A)) - u_1^{-1}(E_f(u_1(A))$.
Since $u_1 \circ u_2^{-1}$ is concave, by applying Corollary 4.5 we have:
$E_f(u_1(A)) = E_f((u_1 \circ u_2^{-1})(u_2(A)) \leq (u_1 \circ u_2^{-1})(E_f(u_2(A)))$.
But u_1^{-1} is increasing, therefore:
$u_1^{-1}(E_f(u_1(A))) \leq u_1^{-1}((u_1 \circ u_2 - 1)(E_f(u_2(A))) = u_2^{-1}(E_f(u_2(A)))$.
By taking into account (a) and the preceding inequality, it follows $\rho_1 \geq \rho_2$.
(3) $\Rightarrow$ (1) Let $x \in \mathbf{R}$. We consider a fuzzy number A and a weighting function such that $x = E_f(A)$. According to Proposition 6.3 we have:
$\rho_{A,f,u_1} = \frac{1}{2} Var_f^*(A) r_1(x)$
$\rho_{A,f,u_2} = \frac{1}{2} Var_f^*(A) r_2(x)$
Since $Var_f^*(A) \geq 0$ and $\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$ it follows $r_1(x) \geq r_2(x)$. ■

The proof of the previous theorem uses the possibilistic Jensen's inequality (Corollary 4.5) and the fact that the possibilistic risk premium is expressed according to the Arrow–Pratt index (Proposition 6.3).

Theorems 8.1 and 8.3 have in common the equivalent conditions (1) and (2). Then by combining Theorems 8.1 and 8.3 we obtain:

Theorem 8.4 *Let u_1, u_2 be two utility functions and $r_1 = r_{u_1}$, $r_2 = r_{u_2}$ the Arrow–Pratt indices of u_1 and u_2. The following assertions are equivalent:*
(1) $r_1(x) \geq r_2(x)$, for any $x \in \mathbf{R}$;
(2) $u_1 \circ u_2^{-1}$ is concave;
(3) For any random variable X with respect to $(\mathbf{R}, \mathbf{B})$, $\rho_{X,u_1} \geq \rho_{X,u_2}$;
(4) For all fuzzy numbers A and weighting functions f, we have:
$\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$.

If condition (4) of Theorem 8.4 holds, then we denote $u_1 \succeq_{possib} u_2$ and we will say that the agent represented by u_1 is more risk–prone to the possibilistic risk than the agent represented by u_2.

Remark 8.5 *The equivalence (3) $\Leftrightarrow$ (4) of Theorem 8.4 gets the form:*
$u_1 \succeq_{probab} u_2$ *iff* $u_1 \succeq_{possib} u_2$.

The significance of this equivalence is remarkable: the probabilistic risk aversion is equivalent with the possibilistic risk aversion.

Remark 8.6 $\succeq_{probab}$ *and* $\succeq_{possib}$ *are two preference relations defined on the set of utility functions. Remark 8.5 says that the restrictions to* $\succeq_{probab}$ *and* $\succeq_{possib}$ *to the set of utility functions which verify the conditions of Proposition 6.3 coincide.*

An open problem is if the equivalence of Remark 8.5 still holds when instead of fuzzy numbers we consider another class of possibilistic distributions.

9 Concluding remarks

The main idea of this paper is to treat the risk aversion by using possibilistic theory instead of probability theory. Concretely, instead of a random variable we consider a possibilistic distribution, and instead of probabilistic indicators we consider possibilistic indicators.

Based on these possibilistic indicators one proposes a possibilistic model of risk aversion.

The main contributions of this paper are:

-the introduction of the notions of possibilistic risk premium and possibilistic relative risk premium, as measures for the risk aversion of an agent represented by the utility function;

-the proof of some formulae for the calculation of the possibilistic risk premium and of the possibilistic relative risk premium, by using the possibilistic expected value and a new indicator which is a version of the possibilistic variance.

-the proof of a possibilistic version of Pratt theorem.

This paper could be the starting point in the elaboration of a possibilistic theory of risk aversion. We mention some topics whose treatment could be tried in this context.

(1) An important problem in risk theory is that of comparing the riskiness distributions (see [22], p. 21). Among numerous contributions, we mention the paper [23] in which Rothschild and Stiglitz prove the equivalence of a number of possible definitions of "increasing risk". The replacement of various types of "stochastic dominance" with appropriate concepts of "possibilistic dominance", the comparison and the application of the latter in the analysis of risk phenomena is an open problem. For example, we could try to obtain some possibilistic versions of the results of Rothschild and Stiglitz from [23].

(2) Another open problem would be to find some notions of generalized possibilistic risk premium which should express the risk aversion of an agent faced with a situation with several risk parameters (expressed by a possibilistic vector $(A_1, \ldots, A_n)$ where each A_i is a fuzzy number). Such a notion would find applicability in evaluating the risk aversion in grid computing [7]. If the functioning of a grid composed by n nodes is described by a possibilistic vector $(A_1, \ldots, A_n)$, then the risk aversion would be evaluated by the generalized possibilistic risk premium.

(3) The enlargement of the framework in the development of the possibilistic risk is another important problem. The replacement of the fuzzy numbers with another class of possibilistic distributions imposes the normalization of the notions of mean value and variance. The use of the definitions from [6], [9], [16], [18], etc. leads to another theory of risk aversion. The value of such a theory is appreciated by means of mathematical results which could be obtained and by its validation in real situations.

The change of the framework can be done also by considering other types of utility functions. Fuzzy utility functions can be one of the candidates to replace the usual utility functions. Among numerous papers which study fuzzy utility functions, we mention only Billot's paper [2].

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References

[1] K. J. Arrow, Rational choice functions and orderings, Economica, 26, 1959, 121–127

[2] A. Billot, An existence theorem for fuzzy utility functions: a new elementary proof, Fuzzy Sets and Systems, 74, 1995, 271–276

[3] C. Carlsson, R. Fullér, On possiblistic mean value and variance of fuzzy numbers. Fuzzy Sets and Systems, 122, 2001, 315–326

[4] C. Carlsson, R. Fullér, Fuzzy reasoning in decision making and optimization. Studies in Fuzziness and Soft Computing Series, Vol. 82, Springer-Verlag, Berlin/Heildelberg, 2002

[5] C. Carlsson, R. Fullér, Peter Majlender, A possibilistic approach to selecting portfolios with highest utility score. Fuzzy Sets and Systems, 131, 2002, 13–21

[6] I. Couso, D. Dubois, S. Montes, L. Sanchez. On various definitions of the variance of a fuzzy random variable. Dans : International Symposium on Imprecise Probability: Theories and Applications (ISIPTA 2007), Prague (Czech Republic), G. De Cooman, J. Vejnarova, M. Zaffalon (Eds.), Sipta, p. 135-144, 2007.

[7] K. Djemane, I. Gourlay, J. Padgett, G. Birkenheuer, M. Novestadt, O. Kao, K. Voß, Introducing risk management into the Grid, Proc. Second IEEE International Conference on e–Science and Grid Computing, 2006

[8] D. Dubois, H. Prade, Fuzzy sets and systems: theory and applications. Academic Press, New York, 1980

[9] D. Dubois, H. Prade, Possibility theory. Plenum Press, New York, 1988

[10] D. Dubois, H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems, 24, 1987, 279–300

[11] R. Fullér, P. Majlender. On weighted possibilistic mean and variance of fuzzy numbers. Fuzzy Sets and Systems, 136, 2003, 363–374

[12] R. Fullér, Introduction to neuro–fuzzy systems. Advances in Soft Computing, Springer–Verlag, Berlin/Heidelberg, 2000

[13] I. Georgescu, Possibilistic risk aversion, submitted

[14] I. Georgescu, Possibilistic risk premium, TUCS Technical Report 852, 2007, Turku, Finland

[15] J. Jacod, P. Protter, Probability essentials, Springer 2000

[16] V. Kreinovich, G. Xiang, S. Ferson, Computing mean and variance under Dempster-Shafer uncertainty: Towards faster algorithms International Journal of Approximate Reasoning, Volume 42, Issue 3, August 2006, Pages 212-227

[17] J. J. Laffont, The economics of uncertainty and information, MIT Press, Cambridge, 1993

[18] B. Liu, Y. K. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE Transactions on Fuzzy Systems, 10, 2002, 445–450

[19] P. Majlender, A normative approach to possibility theory and decision support, Turku Centre for Computer Science PhD Thesis No 54, 2004

[20] M. Mares, Fuzzy cooperative games, Physica Verlag, Heidelberg, 2001

[21] J. Pratt, Risk aversion in the small and in the large, Econometrica, 32, 1964, 122–130

[22] J. Quiggin, Generalized expected utility theory/The Rank-Dependent expected utility model, Kluwer-Nijhoff, Amsterdam, 1993

[23] M. Rothschild, J. Stiglitz, Increasing risk: I. A definition, J. Economic Theory, 2, 1970, 225–243

[24] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems, 1, 1978, 3–28

[25] L. A. Zadeh, Fuzzy sets. Information and Control, 8, 1965, 338–353