Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Approximate Reasoning
43 (2006) 108–112

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

www.elsevier.com/locate/ijar

Discussion

Comments on the paper: "On the properties of equidifferent OWA operator" by Xinwang Liu

Péter Majlender

Institute for Advanced Management Systems Research, Åbo Akademi University, DataCity B 6708, Lemminkäinengatan 14B, FIN-20520 Åbo, Finland

Received 13 December 2005; received in revised form 14 February 2006; accepted 16 February 2006
Available online 29 March 2006

Abstract

In this communication, we will point out an interesting correspondence between a recent paper "On the properties of equidifferent OWA operator" [International Journal of Approximate Reasoning, in press, doi:10.1016/j.ijar.2005.11.003] by Liu, and an earlier result "On obtaining minimal variability OWA operator weights" [Fuzzy Sets and Systems 136 (2003) 203–215] by Fullér and Majlender.

1. Introduction

In [2], Liu proposed a weight-generating method for computing *minimal variability* OWA operator weights under a given level of *orness*. By referring to an earlier result by Fullér and Majlender, published in [1], Liu claimed that with respect to computational efficiency, his approach is superior to the formulation of [1].

In this paper, by using simple reformulations, we will show a fundamental correspondence between the presentations of [1] and [2]. In particular, we shall point out that in part of [2], a *reformulated version* of the results of [1] were only presented.

DOI of original articles: 10.1016/j.ijar.2005.11.003, 10.1016/j.ijar.2006.02.004
E-mail address: peter.majlender@abo.fi

0888-613X/$ - see front matter
doi:10.1016/j.ijar.2006.02.005

In the following, we will briefly summarize the analysis worked out in [1], and then show how it implies all the corresponding results of [2].

2. Summary of the paper: "On obtaining minimal variability OWA operator weights" [Fuzzy Sets and Systems 136 (2003) 203–215]

In [1, p. 205], Fullér and Majlender defined the *variance* of an OWA weighting vector, and formulated a methodology to derive minimal variability OWA operator weights under a given level of compensation, i.e. *orness*. This approach was based on the solution of the following mathematical programming problem [1, p. 205, (2)]:

$$\begin{aligned} \text{minimize} \quad D^2(W) = &\ \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \\ \text{subject to orness } (W) = &\sum_{i=1}^{n}\frac{n-i}{n-1}w_i = \alpha, \quad 0 \leqslant \alpha \leqslant 1, \\ &\sum_{i=1}^{n} w_i = 1, \quad 0 \leqslant w_i \leqslant 1, \ \ i = 1,\ldots,n. \end{aligned} \tag{1}$$

Assuming that the optimal weights represent a window-type OWA aggregation technique [1, p. 206, (4)], from the Kuhn–Tucker second-order sufficiency conditions for optimality, Fullér and Majlender derived the exact optimal solutions to problem (1), as a function of the level of orness α [1, page 208]. In particular, they introduced the following partition of the unit interval [1, pp. 207–208, (9)]

$$(0,1) = \bigcup_{\substack{r,s\in\{1,\ldots,n\} \\ (r-1)(s-n)=0}} J_{r,s} = \bigcup_{r=2}^{n-1} J_{r,n} \cup J_{1,n} \cup \bigcup_{s=2}^{n-1} J_{1,s}, \tag{2}$$

where

$$J_{r,n} = \left(1 - \frac{1}{3}\frac{2n+r-2}{n-1}, 1 - \frac{1}{3}\frac{2n+r-3}{n-1}\right], \quad r = 2,\ldots,n-1, \tag{3}$$

$$J_{1,n} = \left(1 - \frac{1}{3}\frac{2n-1}{n-1}, 1 - \frac{1}{3}\frac{n-2}{n-1}\right), \tag{4}$$

$$J_{1,s} = \left[1 - \frac{1}{3}\frac{s-1}{n-1}, 1 - \frac{1}{3}\frac{s-2}{n-1}\right), \quad s = 2,\ldots,n-1, \tag{5}$$

and pointed out that the form of the optimal weighting vector is fundamentally depends on which subinterval $J_{r,s}$ contains the value of α. Furthermore, based on the selection of r and s ($1 \leqslant r < s \leqslant n$, where either $r = 1$ or $s = n$), such that $\alpha \in J_{r,s}$, Fullér and Majlender proved that the minimal variability OWA operator weights with degree of orness α are computed by [1, p. 208, (10)–(11)]

$$W^* = [w_1^*,\ldots,w_n^*]^{\mathrm{T}} = [0,\ldots,0,w_r^*,\ldots,w_s^*,0,\ldots,0]^{\mathrm{T}}, \tag{6}$$

where

$$w_j^* = 0 \quad \text{if } j < r \quad \text{or} \quad j > s, \tag{7}$$

and

$$w_r^* = \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)}, \tag{8}$$

$$w_s^* = \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)}, \tag{9}$$

and

$$w_j^* = \frac{s-j}{s-r} w_r^* + \frac{j-r}{s-r} w_s^* \quad \text{if } r < j < s. \tag{10}$$

We note that (10) can easily be reformulated as

$$w_j^* = w_r^* + \frac{j-r}{s-r}(w_s^* - w_r^*), \quad r < j < s. \tag{11}$$

In what follows, we shall prove that in Section 3.1, part of Section 3.2 and the Appendix in [2], a *reformulated version* of these results were only presented.

3. Deriving results of the paper "On the properties of equidifferent OWA operator" [International Journal of Approximate Reasoning, in press]

In Section 3.1, Liu defined the concept of *equidifferent OWA operator* by [2, (2)], and by differentiating the cases when the associated weights are monotne decreasing or increasing, he reformulated this definition in [2, (3)] and [2, (4)], respectively. There, a was a constant, d denoted the difference between the adjacent positive weights, and m stood for the number of positive weights[1].

First, from (11) it is obvious that the optimal weights formulated by (7)–(10) define an equidifferent OWA operator with $m = s - r + 1$ and

$$d = \frac{w_s^* - w_r^*}{s-r}.$$

Furthermore, using the definitions of r and s, and the relationships formulated by (2)–(10), we can observe that the case of [2, (3)] discussed and analyzed in [2] is nothing else but a *revisit* of the case $r = 1$, $s \in \{2,\ldots,n\}$ with $\Omega = \alpha \in [0.5, 1)$, $m = s$ and

$$d = \frac{w_s^* - w_r^*}{s-r} = \frac{w_s^* - w_1^*}{s-1} = \frac{6[2(n-1)(1-\alpha) - s + 1]}{(s-1)s(s+1)}$$

(compare this with [2, (12) and (20)]).

Analogously, we can readily see that the case of [2, (4)] analyzed in detail in [2] is a *reformulation* of the case $r \in \{1,\ldots,n-1\}$, $s = n$ with $\Omega = \alpha \in (0, 0.5]$, $m = n - r + 1$ and

$$d = \frac{w_s^* - w_r^*}{s-r} = \frac{w_n^* - w_r^*}{n-r} = \frac{6[2(n-1)(1-\alpha) - n - r + 2]}{(n-r)(n-r+1)(n-r+2)}$$

(compare this with [2, (17) and (20)]).

[1] Actually, m denoted the number of equidifferent elements among the weights, i.e. the number of weights that are part of an arithmetic progression. However, it is nothing else but the number of positive weights, which is increased by one in the case d is a divisor of a, i.e. $d|a$.

In particular, if $r = 1$ and $s = n$, then

$$d = \frac{w_s^* - w_r^*}{s - r} = \frac{w_n^* - w_1^*}{n - 1} = \frac{6(1 - 2\alpha)}{n(n + 1)}$$

(compare this with [2, (20)]).

Hence, in Section 3.1, [2] some results of [1] were only presented by introducing the notations $m = s$ if $\Omega = \alpha \in [0.5, 1)$, and $m = n - r + 1$ if $\Omega = \alpha \in (0, 0.5]$.

In Section 3.2, when deriving the *maximum spread* equidifferent OWA operators, Liu computed the values of m by using expressions [2, (18)–(19)]. However, it was not admitted that [2, (18)–(19)] actually presented a *reformulation* of the relationship

$$\alpha \in J_{r,s}$$

in the context of [1], solved for s (when $r = 1$) and $n - r + 1$ (when $s = n$). Indeed, from (5) we obtain for the case $\alpha \in [0.5, 1)$:

$$\begin{aligned} \alpha \in J_{1,s} &\iff 1 - \frac{1}{3}\frac{s-1}{n-1} \leqslant \alpha < 1 - \frac{1}{3}\frac{s-2}{n-1} \\ &\iff 3(n-1)(1-\alpha) + 1 \leqslant s < 3(n-1)(1-\alpha) + 2 \\ &\iff s = \lfloor 3(n-1)(1-\alpha) + 2 \rfloor \end{aligned}$$

(since $s \in \mathbb{N}$), where $\lfloor \cdot \rfloor$ denotes the floor function, which corresponds with parts of [2, (18)–(19)] with $\Omega = \alpha$ and $m = s$. Analogously, from (3) we get for the case $\alpha \in (0, 0.5]$:

$$\begin{aligned} \alpha \in J_{r,n} &\iff 1 - \frac{1}{3}\frac{2n+r-2}{n-1} < \alpha \leqslant 1 - \frac{1}{3}\frac{2n+r-3}{n-1} \\ &\iff n - 1 - 3(n-1)\alpha < r \leqslant n - 3(n-1)\alpha \\ &\iff n - r + 1 = \lfloor 3(n-1)\alpha + 2 \rfloor \end{aligned}$$

(since $n - r + 1 \in \mathbb{N}$), which corresponds with the remaining parts of [2, (18)–(19)] with $\Omega = \alpha$ and $m = n - r + 1$.

Thus, in part of Section 3.2, [2] some other results of [1] were only presented again by using the notations $m = s$ if $\Omega = \alpha \in [0.5, 1)$, and $m = n - r + 1$ if $\Omega = \alpha \in (0, 0.5]$.

In particular, [2, (18)–(19)] actually formulated the *same interval partition* as the one presented in [1], i.e. (2)–(5). Furthermore, by using the relationships between r, s and m specified above, we can easily verify that formulas [2, (21)–(22)], as well as the algorithm based on them, represent a *summary* of the method introduced in [1]. Especially, the argument following Theorem 2, claiming that the newly proposed method is superior to the solution of [1] is inappropriate.

After observing these similarities, it is straightforward to see that the proof of Theorem 2 in the Appendix merely *recomputes* the proof presented in [1, pp. 210–211, (i)]. That is, selecting the case $r = 1$, $s \in \{2, \ldots, n\}$, from [1, pp. 210–211, (i)] we indeed obtain the whole proof of [2, Appendix] with $m = s$, $\Omega = \alpha$, and $\lambda_1 = \lambda_2^*$, $\lambda_2 = \lambda_1^*$, and

$$a = w_r^* = w_1^*, \quad d = \frac{w_s^* - w_r^*}{s - r} = \frac{w_s^* - w_1^*}{s - 1} = \frac{6[2(n-1)(1-\alpha) - s + 1]}{(s-1)s(s+1)}.$$

4. Conclusions

In this communication, we pointed out a fundamental correspondence between the results presented in [1] and parts of [2]. In particular, we showed that in Section 3.1, part of Section 3.2 and the Appendix in [2], a *reformulated version* of the results of [1] were only presented. By using appropriate reformulations, we proved that the method of [2] was based on the *same partition of the unit interval* and *weight-generating formulas* as the ones developed in [1].

References

[1] R. Fullér, P. Majlender, On obtaining minimal variability OWA operator weights, Fuzzy Sets and Systems 136 (2003) 203–215.

[2] X. Liu, On the properties of equidifferent OWA operator, International Journal of Approximate Reasoning, in press, doi:10.1016/j.ijar.2005.11.003.